

Anomalous is Ubiquitous

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HIT

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A panoramic tour of the scenic mountainous terrain of the Diffusion Kingdom

Joint research with *Joseph Klafter* TAU

Brownian Diffusion

Diffusion

- The most elemental random transport processes in Science and Engineering are **diffusions**
- The archetypal model of diffusion processes is **Brownian motion (BM)**
- **BM** is applied in a host of scientific fields – from Physics and Chemistry to Biology and Finance

Van Kampen (2007), Schuss (2009), ...

Brownian Landmarks

- **Discovery** (Brown 1827)
- **Financial modeling** (Bachelier 1900)
- **Diffusion modeling** (Einstein-Smoluchowski 1905)
- **Mathematical construction** (Wiener 1923)
- **Stochastic calculus** (Ito 1940s)
- **Geometric BM** (Samuelson 1965)
- **Option pricing** (Merton-Black-Scholes 1973)

Brownian Motion

Brownian motion (BM) is a random process whose increments are

- independent, stationary, and Gaussian

Brownian noise (BN) – the ‘discrete derivative’ of BM – is a sequence of

- i.i.d. Gaussian random variables

Brownian Universality

Random Walks:

$$W(t) = Z_1 + \cdots + Z_{[t]} \quad (t \geq 0)$$

Z s i.i.d. with zero mean and finite variance

Scaling:

$$W(t) = (1/\sqrt{n})W(nt)$$

Functional CLT:

Brownian motion is the universal stochastic scaling limit of random walks (Donsker 1951)

**Macroscopic BM statistics emerge
invariantly with respect to the
microscopic RW statistics**

**BM reigned supreme the Diffusion
Kingdom for almost a century**

Anomalous Diffusion

**In recent decades non-Brownian
upheavals have been quaking
the Diffusion Kingdom with
ever increasing intensity**

Bouchaud-Georges 1990,
Metzler-Klafter 2000, Klafter-Sokolov 2005, ...

Mean Square Displacements

Brownian motion MSD:

$$E[|B(t)|^2] = ct$$

Brownian noise MSD:

$$E[|\Delta B(t)|^2] = c$$

'Anomalous' MSD:

$$E[|\xi(t)|^2] = ct^\epsilon$$

Mean Square Displacements

Sub diffusion:

$$0 < \epsilon < 1$$

- charge transport in amorphous semiconductors
(Scher-Montroll 1975)
- contaminants propagation in ground water
(Kirchner-Feng-Neal 2006)
- protein motion in intracellular media
(Golding-Cox 2006) ...

Mean Square Displacements

Super diffusion:

$$\epsilon > 1$$

- tracer transport in turbulent flows
(Solomon-Weeks-Swinney 1993)
- search patterns of foraging animals
(Sims et. al. 2008)
- particle trajectories in plasma
(Liu-Goree 2008) ...

Auto Correlations

Brownian noise AC:

$$\text{Cov}[\Delta B(t), \Delta B(t+1)] = 0$$

'Anomalous' AC:

$$\text{Cov}[\xi(t), \xi(t+1)] \approx c/l^\varepsilon$$

Long-range dependence:

$$0 < \varepsilon < 1$$

The “Joseph effect” (Mandelbrot-Wallis 1968)

Auto Correlations

Long-range dependence:

- human walk (Hausdorff et. al. 1995)
- sedimentation (Segre et. al. 1997)
- atmospheric processes (Bunde et. al. 1998)
- price volatility (Gopikrishnan et. al. 1999)
- heartbeat dynamics (Ivanov et. al. 2001)
- seismic coda (Campillo-Paul 2003) ...

Power Spectra

Brownian motion PS:

$$S_B(f) = c/|f|^2$$

Brownian noise PS:

$$S_{\Delta B}(f) = c$$

'Anomalous' PS:

$$S_{\xi}(f) = c/|f|^{\epsilon}$$

Power Spectra

- Power-law power spectra are termed:
“1/f noise”
- 1/f noise is ubiquitous – over 1400 citations in
<http://www.nslj-genetics.org/wli/1fnoise>
- Typical exponent range:
 $0 < \epsilon < 2$
BN ‘boundary’ $\epsilon=0$, BM ‘boundary’ $\epsilon=2$

Marginal Distributions

Brownian motion/noise MD:

Gaussian – characterized by Fourier transform

$$F_B(\theta) = \exp(-c |\theta|^2)$$

'Anomalous' MD:

Levy – characterized by Fourier transform

$$F_\xi(\theta) = \exp(-c |\theta|^\varepsilon)$$

Exponent range:

$$0 < \varepsilon < 2$$

Marginal Distributions

Gaussian tails – super-exponential decay:

$$\Pr(|B(t)| > l) \approx \exp(-cl^2)$$

Levy ‘heavy tails’ – power-law decay:

$$\Pr(|\xi(t)| > l) \approx c/l^\epsilon$$

The “Noah effect” (Mandelbrot-Wallis 1968)

Marginal Distributions

Levy marginal distributions:

- anomalous transport (Shlesinger et. al. 1993)
- plasma dynamics (Chechkin et. al. 2002)
- solar wind (Watkins et. al. 2005)
- bill trajectories (Brockmann et. al. 2006)
- search processes (Condamin et. al. 2007)
- light scattering (Barthelemy et. al. 2008) ...

**‘Anomalous’ stochastic phenomena
as well as statistical selfsimilarity
was widely observed also in
internet-age data traffic**

Leland et. al. 1994, Paxon-Floyd 1994,
Crovella-Bestavros 1996, Willinger et. al. 1997, ...

Fractional Brownian Diffusion

Fractional Brownian Motion

Fractional Brownian motion (FBM) is a random process whose increments are

- dependent, stationary, and Gaussian

Fractional Brownian noise (FBN) – the ‘discrete derivative’ of FBM – is a sequence of

- stationary Gaussian random variables

(Mandelbrot-Van Ness 1968)

Fractional Brownian Motion

FBM's is selfsimilar with Hurst exponent: $0 < H < 1$

➤ $0 < H < \frac{1}{2}$

negative correlation – anti-persistence

➤ $H = \frac{1}{2}$

zero correlation – Brownian motion

➤ $\frac{1}{2} < H < 1$

positive correlation – persistence

'Anomalous' Behavior

Mean Square Displacement:

- anti-persistence => FBM sub-diffusion
- persistence => FBM super-diffusion

Correlations:

- persistence => FBN long-range dependence

Power Spectrum:

- anti-persistence/persistence => FBM 1/f noise

Attaining FBM

FBM is attained as the stochastic scaling limit of superpositions of i.i.d. random processes:

- renewal processes (Mandelbrot 1969)
- correlated random walks (Davydov 1970)
- on-off processes (Taqqu et. al. 1997)
- OU processes (Leonenko-Tauber 2005) ...

**In the superposition models yielding FBM
a pre-set target output (FBM) is attained
from specific classes of input processes**

Signal Superposition Model

Signal Superposition Model

- A multitude of transmission sources
- Source k transmits the signal pattern

$$\mathbf{X}_k = (\mathbf{X}_k(t))_{t \geq 0}$$

- The transmission parameters of source k are:
 - amplitude \mathbf{a}_k
 - frequency ω_k
 - initiation epoch τ_k

Signal Superposition Model

➤ The output process

$$Y = (Y(t))_{t \geq 0}$$

is the superposition of all source-transmissions:

$$Y(t) = \sum a_k X_k(\omega_k(t - \tau_k))$$

Model Assumptions

- The signal patterns $\{X_k\}_k$ are i.i.d. copies of a general and arbitrary signal pattern

$$X = (X(t))_{t \geq 0}$$

- The transmission parameters $\{(a_k, \omega_k, \tau_k)\}_k$ are a Poisson process with general intensity function

$$\lambda(a, \omega, \tau)$$

- The signal patterns and transmission parameters are mutually independent

Model Assumptions

Why Poisson?

- In large systems it is natural to assume that the transmission parameters $\{(a_k, \omega_k, \tau_k)\}_k$ are random
- The common modeling of random scattering of points in general domains is Poissonian
- Poisson processes have a wide spectrum of applications in Science and Engineering

(Kingman 1993, Wolff 1989, Embrechts et. al. 1997)

**The signal superposition model
is general and accommodates
a span of stochastic models**

$$Y(t) = \sum a_k X_k(\omega_k(t - \tau_k))$$

Random Walks:

- Signal patterns: $X_k(t) = Z_k$
 $\{Z_k\}_k$ i.i.d. with zero mean and finite variance
- Amplitudes: $a_k = 1/\sqrt{n}$
- Frequencies: irrelevant
- Initiation epochs: Poisson process with rate n

$$Y(t) = \sum a_k X_k(\omega_k(t - \tau_k))$$

Shot Noise processes:

- Signal patterns: $X_k(t) = \exp(-t)$
- Amplitudes: i.i.d. random variables
- Frequencies: $\omega_k = r$
 r relaxation constant
- Initiation epochs: Poisson process with unit rate

(Rice 1944-45, Lowen-Teich 1989-90, Eliazar-Klafter 2007)

$$Y(t) = \sum a_k X_k(\omega_k(t - \tau_k))$$

M/G/ ∞ processes:

➤ Signal patterns: $X_k(t) = \chi(0 < t < 1)$

➤ Amplitudes: $a_k = 1$

➤ Frequencies: $\omega_k = 1/T_k$

$\{T_k\}_k$ i.i.d. sojourn times

➤ Initiation epochs: Poisson process with unit rate

(Takacs 1962, Cox-Miller 1965, Cox-Isham 1980)

The signal superposition model

$$Y(t) = \sum a_k X_k(\omega_k(t-\tau_k))$$

**is a generalized version of the
Einstein-Smoluchowski
diffusion model**

Universality

Universality Quest

Considering a specific statistic $\Phi(\mathbf{Y})$ of the output process \mathbf{Y} of the signal superposition model:

- Are there Poissonian parameter-randomizations that render the output-statistic $\Phi(\mathbf{Y})$ invariant with respect to the input signals $\{\mathbf{X}_k\}_k$?
- If yes, then what is the form of the resulting input-invariant output-statistic $\Phi(\mathbf{Y})$?

Mean Square Displacements

$\Phi(Y)$ is the MSD of the output process

Theorem: Poissonian parameter-randomizations that render the output MSD invariant with respect to the input signal patterns exist and yield the power-law MSD

$$E[|Y(t)|^2] = c_X t^\varepsilon$$

with exponent $\varepsilon > 0$

Auto Correlations

$\Phi(Y)$ is the AC of stationary output processes

Theorem: Poissonian parameter-randomizations that render the output AC invariant with respect to the input signal patterns exist and yield the power-law AC

$$\text{Cov}[Y(t), Y(t+l)] = c_x / l^\varepsilon$$

with exponent $0 < \varepsilon < 1$

Power Spectra

$\Phi(Y)$ is the PS of the output process

Theorem: Poissonian parameter-randomizations that render the output PS invariant with respect to the input signal patterns exist and yield the power-law PS

$$S_Y(f) = c_X / |f|^\epsilon$$

with exponent $0 < \epsilon < 2$

Marginal Distributions

$\Phi(Y)$ is the MD of the output process

Theorem: Poissonian parameter-randomizations that render the output MD invariant with respect to the input signal patterns exist and yield the Fourier transform

$$E[\exp(i\theta Y(t))] = \exp(-c_x |\theta|^\varepsilon)$$

with exponent $0 < \varepsilon < 2$

MAGIC

Statistic considered:

- Mean Square Displacement
- Auto Correlation
- Power Spectra
- Marginal Distributions

Invariance result:

- Sub diffusion and super diffusion
- Long-range dependence
- $1/f$ noise
- Levy laws

MAGIC

- **Perfect coincidence between ‘anomalous’ empirical observations and theory**
- **Universal generation of ‘anomalous’ behavior via a single model and one approach**
- **‘Anomalous’ behavior emergent rather than pre-set as desired goal**
- **Robustness and resilience with respect to ‘environmental changes’**

**Einstein-Smoluchowski perspective:
The only macroscopic statistics which are
invariant with respect to microscopic
statistics are ‘anomalous’!**

Sources

Iddo Eliazar & Joseph Klafter:

- PNAS 106 (2009) 12251
- J. Phys. A 42 (2009) 472003
- J. Phys. A 43 (2010) 132002
- Phys. Rev. E 82 (2010) 021109

Ultra Diffusion

Ultra Diffusion

BM characterization:

- Variance characterization of BM:

$$\text{Var}[B(t)] = c \cdot t$$

- Fourier characterization of BM:

$$E[\exp(i\theta B(t))] = \exp(-c \cdot t \cdot |\theta|^2 / 2)$$

- Fourier characterization does not require a finite-variance condition to hold

Ultra Diffusion

Definition:

A stochastic process $(\xi(t))_{t \geq 0}$ is said to be an ultra diffusion if

$$E[\exp(i\theta\xi(t))] = \exp(-c \cdot \psi(t) \cdot \phi(\theta))$$

In the finite-variance case

$$\text{Var}[\xi(t)] = (c\phi''(0)) \cdot \psi(t)$$

Ultra Diffusion

Examples:

- Poisson and compound Poisson processes
- Brownian motion and FBM
- Stable Levy motions and FSLM
- General Levy processes
- OU processes driven by stable Levy noises
- M/G/ ∞ processes

Ultra Diffusion

Theorem: Poissonian parameter-randomizations that render the output process **Y** an ultra diffusion, invariantly with respect to the input signal patterns $\{\mathbf{X}_k\}_k$, exist and yield the Fourier transform

$$E[\exp(i\theta Y(t))] = \exp(-c_x \cdot t^\alpha \cdot |\theta|^\beta)$$

with exponents $\alpha > 0$ and $0 < \beta < 2$

Eliazar-Klafter, J. Phys. A 43 (2010) 132002

Ultra Diffusion

Conclusions:

- Joint universal emergence of 'sub/super diffusion' and Levy laws
- Intrinsic scaling (equality in law):

$$Y(t) = t^{\alpha/\beta} \cdot Y(1)$$

- Probability density at the origin:

$$\Pr(Y(t)=0) = \Pr(Y(1)=0) / t^{\alpha/\beta}$$

Self Similarity

Self Similarity

Definition:

A stochastic process $(\xi(t))_{t \geq 0}$ is said to be selfsimilar with Hurst exponent H if the scaled processes

$$(Y(st))_{t \geq 0} \text{ and } (s^H Y(t))_{t \geq 0}$$

are equal in law (for any positive scale s)

Self Similarity

Examples:

- Brownian motion
- Fractional Brownian motion
- Stable Levy motions
- Fractional Stable Levy motions

Self Similarity

Theorem:

The output process \mathbf{Y} is selfsimilar, with Hurst exponent H , invariantly of the input signal patterns $\{\mathbf{X}_k\}_k$, if and only if the Poissonian intensity satisfies the scaling relation:

$$\lambda(s^H a, \omega/s, s\tau) = \lambda(a, \omega, \tau)/s^H$$

Eliazar-Klafter, Phys. Rev. Lett. 103 (2009) 040602

Finite-Variance Corollaries

- Selfsimilarity => sub/super diffusion:

$$E[|Y(t)|^2] = c_X t^{2H}$$

- Selfsimilarity => 1/f noise:

$$S_Y(f) = c_X / |f|^{1+2H}$$

- Selfsimilarity, stationary increments, $\frac{1}{2} < H < 1$
=> long-range dependence of the 'derivative':

$$\text{Cov}[\Delta Y(t), \Delta Y(t+l)] \approx c_X / l^{2-2H}$$

Beyond Diffusion

The concept of input-invariant Poissonian randomizations is extremely powerful and is applicable well beyond the Diffusion Kingdom

Propagating Populations

Populations of propagating particles in the d -dimensional Euclidean space:

$$Y_k(t) = a_k X_k(\omega_k(t - \tau_k)) \quad (t \geq \tau_k)$$

- Invariance of the particles' displacements with respect to the spatial trajectories $\{X_k\}_k$
- Invariance of the particles' first passage times with respect to the spatial trajectories $\{X_k\}_k$

Search Swarms

Swarms of agents in a general topological space:

$$Y_k(t) = X_k(\omega_k(t - \tau_k)) \quad (t \geq \tau_k)$$

- The agents are observed only when in a target zone – a general subset of space
- Invariance of the agents' 'target-zone statistics' with respect to the spatial trajectories $\{X_k\}_k$

Randomized CLTs

The **stable Levy laws** and the **Extreme Value laws** are the universal stochastic limit laws of sums and extrema of i.i.d. random variables

CLT approach:

- deterministic and uniform scaling
- narrow domains of attraction

RCLT approach:

- Poissonian random and non-uniform scaling
- wide domains of attraction

Sources

Iddo Eliazar & Joseph Klafter:

- Chem. Phys. 370 (2010) 290
- Phys. Rev. E 82 (2010) 011112
- Phys. Rev. E 82 (2010) 021122
- J. Phys. A 44 (2011) 222001

**The Diffusion Kingdom is vast, and
we just spent a one-hour tour of
the ‘anomalous’ peaks
surrounding its
Brownian
center**

The End

Thank you